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Effective Field Theory for the Crystalline Colour Superconductive Phase of QCD

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Abstract

We present an effective field theory for high density, low temperature QCD in the crystalline colour superconductive phase (*LOFF* phase). This interesting phase of QCD is characterized by a gap parameter with a crystalline pattern, breaking traslational and rotational invariance, and could have astrophysical applications. In the effective theory the fermions have a Majorana mass, which, besides colour, breaks $SU(2)_{L,R}$ as well as translation and rotation symmetries. Fermions couple to seven Nambu-Godstone Bosons (NGB) arising from the breaking of rotation and translation invariance (phonon) and $SU(2)_{L,R}$. Integrating out the fermions leads eventually to an effective lagrangian in terms of the NGB fields only .

1 Introduction

We shall consider massless quarks of three colours and two different flavours. At zero density the theory is invariant under the group $SU(3)_c \times SU(2)_L \times$

$SU(2)_R$. We shall disregard in the sequel further invariances under $U(1)$, not essential for the following derivations. The model we shall consider is based on the approximation: $m_1 = m_u \approx 0,\ m_2 = m_d \approx 0$ and large m_s, of the full QCD theory with three flavours.

At higher densities we introduce two chemical potentials, μ_1 and μ_2 for the two species, with (assuming $\mu_1 \geq \mu_2$)

$$\mu_1 - \mu_2 = \delta\mu \ll \mu = \frac{\mu_1 + \mu_2}{2} \ . \tag{1}$$

Physical situations with $\delta\mu \neq 0$ are the most likely to occur in nature. For example, the so-called glitches of neutron stars [1] (see also [2]) should be studied in this hypothesis. More generally, it is theoretically important, independently of applications, to study in detail each single phase of the QCD phase-diagram, which implies the analysis for generic values of $\delta\mu$. This note is devoted to the construction of an effective theory for the physically interesting case $\delta\mu \neq 0$. Before describing our formalism, let us start however with a brief review of the $\delta\mu = 0$ case.

QCD in the limit $\delta\mu = 0,\ \mu \to \infty,\ T = 0$ has been first studied in [3] and subsequently by a number of authors [4], who have shown that a phenomenon of colour superconductivity takes place. Effects of this phase should most likely be seen in astrophysical contexts, for instance in connection with magnetic fields in quark matter [5], quark star oscillations [6], cooling of neutron stars [7], and supernova neutrinos [8].

The origin of the superconducting (BCS) phase can be traced back to the vacuum expectation value of the operator

$$\psi^T_{i\alpha} C \psi_{j\beta} \ , \tag{2}$$

($\alpha,\ \beta = 1, 2, 3$ colour indices; $i,\ j = 1, 2$ flavour indices) that, for μ sufficiently large, is non vanishing. If ψ is used to represent a left-handed two-component Weyl spinor and $C = i\sigma_2$, one gets indeed:

$$\epsilon^{\alpha\beta\gamma}\epsilon_{ij} < \psi^T_{i\alpha} C \psi_{j\beta} >= \Delta\delta^{\gamma 3} \ . \tag{3}$$

The right handed field satisfies the same relation with $\Delta \to -\Delta$ (we assume $\Delta > 0$). The condensate (3) breaks the original symmetry group $SU(3)_c \otimes SU(2)_L \otimes SU(2)_R$ down to

$$SU(2)_c \times SU(2)_L \times SU(2)_R \ . \tag{4}$$

The chiral group remains unbroken, while the original colour symmetry group is broken to $SU(2)_c$, with generators T^A corresponding to the generators T^1, T^2, T^3 of $SU(3)_c$. Therefore three gluons remain massless whereas the remaining five acquire a mass. There are no Nambu-Goldstone Bosons. The effective theory to describe this situation has been discussed in [9] (the three massless flavor case was studied in [10]).

The microscopic dynamics results from a mechanism analogous to the formation of an electron Cooper pair in a BCS superconductor. At $T = 0$ the only QCD interactions are those involving fermions near the Fermi surface. Quarks inside the Fermi sphere cannot interact because of the Pauli principle, unless the interactions involve large momentum exchanges. In this way the quarks can escape the Fermi surface, but these processes are disfavoured, as large momentum transfers imply small couplings due to the asymptotic freedom property of QCD. Even though interactions of fermions near the Fermi surface involve momenta of the order of μ, their effects are not necessarily negligible. As a matter of fact, even a small attractive interaction between fermions near the Fermi surface and carrying opposite momenta can create an instability and give rise to coherent effects. This is what really happens [3] and the result is the formation of a diquark condensate, as expressed by (3). We stress again that the only relevant fermion degrees of freedom are therefore those near the Fermi surface.

This picture holds also for $\delta\mu \neq 0$ provided $\delta\mu \ll \Delta$. On the other hand, for $\delta\mu \approx \Delta$, the picture changes significantly. The analysis in [1] shows that there exist two values of $\delta\mu$, $\delta\mu_1$ and $\delta\mu_2$, such that, for

$$\delta\mu \in (\delta\mu_1, \delta\mu_2) \ , \tag{5}$$

the high density quark-gluon matter is in a phase characterized by the breaking of translational and rotational invariance, due to the presence of a scalar and a vector condensate. This phenomenon is called crystalline colour superconductivity of QCD and the relative phase is named *LOFF* phase, from the initials of the authors [11] who have studied a similar phase in quantum electrodynamics. The *LOFF* phase, is energetically favoured as compared to the BCS phase if (5) holds. The authors of ref. [1] find $\delta\mu_1 = 0.71\Delta$ and $\delta\mu_2 = 0.744\Delta$ for $\mu = 0.4\ GeV$ and $\Delta = 40\ MeV$.

The aim of the present note is to present an effective lagrangian approach to this phase. By integrating out the negative energy fermion fields and the gluons and introducing velocity-dependent positive energy quark fields we shall first obtain an effective theory where the effective quark fields are basically free, but possess a Majorana mass term breaking translational and rotational invariance as well as $SU(2)_{L,R}$ symmetry. This term arises from the presence of a scalar and a vector condensate, as discussed in [1]. This lagrangian represents an extension of the results of [12] to the crystalline colour superconductive phase. We shall then couple the theory to external fields describing the Nambu-Goldstone Bosons arising by the breaking of the above mentioned symmetries. There are 7 NGB. Four of them are associated to $SU(2)_L$ and $SU(2)_R$ breaking. Two of the remaining NGB are associated with the breaking of the rotational invariance, while the last one is the phonon field associated with the breaking of the translational invariance. The integration of the fermion degrees of freedom eventually leads to an effective theory containing as dynamical fields only the NGB fields. We will conclude this note by classifying the possible terms of the resulting effective lagrangian.

2 Effective theory near the Fermi surface

To start with, we extend the approach of Ref. [12] to the two flavour case. We allow $\delta\mu \neq 0$. If p_j is the momentum of the quark having flavour j, we write:

$$p_j = \mu_j v + \ell_j \ . \tag{6}$$

Here $v^\mu = (0, \vec{v})$, where $\vec{v}$ is the Fermi velocity. Since we wish to describe excitations near the Fermi surface, we shall limit the functional integration in the generating functional to fermion fields satisfying [13]

$$|\ell_j| < \mu_j \ . \tag{7}$$

Therefore, introducing fields $\psi_\pm$, corresponding to positive and negative energies for the massless left-handed Weyl fermion of flavour j in the chemical potential μ_j, we have

$$\begin{aligned} \psi(x) &= \sum_{\vec{v}} e^{-i\mu_j v\cdot x} \int_{|\ell_j|<\mu_j} \frac{d^4\ell_j}{(2\pi)^4} e^{-i\ell_j\cdot x}\, \psi_{\vec{v}}(\ell_j) = \\ &= \sum_{\vec{v}} e^{-i\mu_j v\cdot x} \left[\psi_+(x) + \psi_-(x)\right], \end{aligned} \tag{8}$$

where

$$\psi_\pm(x) = \frac{1 \pm \vec{\alpha}\cdot\vec{v}}{2} \int_{|\ell|<\mu_j} \frac{d^4\ell}{(2\pi)^4} e^{-i\ell\cdot x} \psi_{\vec{v}}(\ell) \ . \tag{9}$$

Here $\sum_{\vec{v}}$ means an average over the Fermi velocities and

$$\psi_\pm(x) \equiv \psi_{\pm,\vec{v}}(x) \tag{10}$$

are velocity-dependent fields. The meaning of this decomposition is that the negative energy fields ψ_- can be expressed in terms of the positive energy ones through the formula

$$\psi_- = -\frac{1}{2\mu_j}\gamma_0 \ \not{\partial}_T \psi_+ \ , \tag{11}$$

which results from the equations of motion, and therefore can be integrated out. Here

$$\begin{aligned} \not{\partial}_T &= \partial_\mu \gamma_T^\mu && (12)\\ \gamma_T^\mu &= \gamma_\lambda \left(2g^{\mu\lambda} - V^\mu \tilde{V}^\lambda - V^\lambda \tilde{V}^\mu \right) && (13) \end{aligned}$$

and

$$\begin{aligned} V^\mu &= (1, +\vec{v}) \\ \tilde{V}^\mu &= (1, -\vec{v}) \ . \end{aligned} \tag{14}$$

Eliminating ψ_- results in an effective theory which at the lowest order is described by the lagrangian [12]:

$$\mathcal{L}_0 \ = \ \sum_{\vec{v}} \left(\psi_+^\dagger i V^\mu \partial_\mu \psi_+ \ - \ \frac{1}{2\mu_j} \psi_+^\dagger \left(\not{\partial}_T\right)^2 \psi_+ \right) \ + \ (L \to R) \ . \tag{15}$$

The term $\propto 1/\mu_j$ is in general non leading and can be neglected, which we will do in the sequel[1]. We note explicitly that the fields appearing in this

[1] In [12] it has been shown that it contributes to the Meissner gluon mass in the colour flavour locking phase of QCD [14]. Since we are not interested here in the calculation of the gluon mass arising from the colour breaking of 2-flavour QCD we can neglect it altogether; for a more complete discussion on this see [15].

equation depend on the same velocity $\vec{v}$ because of the Riemann Lebesgue lemma. Off-diagonal contributions would have the form:

$$\sum_{\vec{v}_1\neq\vec{v}_2} e^{i\mu_j(\vec{v}_1-\vec{v}_2)\cdot\vec{x}}\psi^\dagger_{+,\vec{v}_2} iV\cdot\partial\,\psi_{+,\vec{v}_1} + ... \tag{16}$$

that give negligible contributions in the large chemical potential limit, due to the rapid oscillations of the exponential function.

It is useful to use a different basis for the fermion fields. We introduce:

$$\begin{aligned}
\psi_{+,i\alpha} &= \sum_{A=0}^{3}\frac{(\sigma_A)_{i\alpha}}{\sqrt{2}}\varphi_+^A \qquad (i,\,\alpha=1,\,2)\\
\psi_{+,13} &= \varphi_+^4\\
\psi_{+,23} &= \varphi_+^5\ ,
\end{aligned}\tag{17}$$

where σ_A are the Pauli matrices for $A=1,2,3$ and $\sigma_0=1$. Here clearly φ_+^A are positive energy, velocity dependent fields:

$$\varphi_+^A \equiv \varphi_{+,\vec{v}}^A\ . \tag{18}$$

We also introduce

$$\varphi_-^A \equiv \varphi_{+,-\vec{v}}^A\ . \tag{19}$$

$\varphi_\pm^A$ should not be confused with the positive and negative energy fields; they are both positive energy fields, but are relative to opposite velocities.

The lagrangian $\mathcal{L}_0$ can therefore be written as follows:

$$\mathcal{L}_0 \;=\; \sum_{\vec{v}}\sum_{A=0}^{5}\varphi_+^{A\dagger}(iV\cdot\partial)\,\varphi_+^A \;+\; (L\to R)\ . \tag{20}$$

Using the fact that the average over velocities is symmetric we write:

$$\mathcal{L}_0 = \frac{1}{2}\sum_{\vec{v}}\sum_{A=0}^{5}\left(\varphi_+^{A\,\dagger}(iV\cdot\partial)\,\varphi_+^A + \varphi_-^{A\,\dagger}(i\tilde{V}\cdot\partial)\,\varphi_-^A\right)\ . \tag{21}$$

Introducing now

$$\chi^A \;=\; \left(\begin{array}{c}\varphi_+^A\\ \\ C\varphi_-^{A\,*}\ ,\end{array}\right) \tag{22}$$

the lagrangian can be written as follows:

$$\mathcal{L}_0 \;=\; \frac{1}{2}\sum_{\vec{v}}\sum_{A=0}^{5}\chi^{A\,\dagger}\left(\begin{array}{cc} iV\cdot\partial & 0\\ 0 & i\tilde{V}\cdot\partial\end{array}\right)\chi^A\ . \tag{23}$$

3 Crystalline colour superconductive phase

As shown in [1] the vacuum state is characterized by a non vanishing expectation value of a lagrangian term breaking translational and rotational invariance. The appearance of this condensate is a consequence of the fact that for $\mu_1 \neq \mu_2$, and in a given range of $|\mu_1 - \mu_2|$ [1], the formation of a Cooper pair with a total momentum

$$\vec{p}_1 + \vec{p}_2 = 2\vec{q} \tag{24}$$

is energetically favoured in comparison with the normal BCS state, if (5) is satisfied. There are actually two condensates, one with the two quarks in a spin zero state (scalar condensate) and the other one characterized by total spin 1 (vector condensate). In the BCS state the quarks forming the Cooper pair have necessarily $S = 0$; as a matter of fact, since the quarks have opposite momenta and equal helicities, they must be in an antisymmetric spin state. This not true if the total momentum is not zero and the two quarks can have both $S = 0$ and $S = 1$.

The possible form of these condensates is discussed in [1]; in particular the geometric properties of the crystalline phase have been assumed to arise from a simple plane wave behaviour of the condensate $\propto e^{2i\vec{q}\cdot\vec{x}}$, though more complicated structures are possible. We shall limit our analysis to the simplest case; to begin with, we consider the lagrangian term relative to the scalar condensate. We shall write it as follows:

$$\mathcal{L}_\Delta^{(s)} = -\frac{\Delta^{(s)}}{2}\, e^{2i\vec{q}\cdot\vec{x}} \epsilon^{\alpha\beta 3} \epsilon_{ij} \psi^T_{i\alpha}(x) C \psi_{i\beta}(x) \; - (L \to R) + \text{h.c.}\;. \tag{25}$$

Here $\psi(x)$ are positive energy left-handed fermion fields. We introduce velocity dependent positive energy fields $\psi_{+,\,\vec{v}_i;\,i\alpha}$ having flavour i; neglecting the non-leading negative-energy fields we have:

$$\begin{aligned} \mathcal{L}_\Delta^{(s)} \;=\; & -\frac{\Delta^{(s)}}{2}\, e^{2i\vec{q}\cdot\vec{x}} \times \\ & \sum_{\vec{v}_1,\vec{v}_2} \epsilon_{ij}\epsilon^{\alpha\beta 3} e^{i\mu(\vec{v}_1+\vec{v}_2)\cdot\vec{x} + i\frac{\delta\mu}{2}(\vec{v}_1-\vec{v}_2)\cdot\vec{x}} \psi_{+,\,\vec{v}_i;\,i\alpha}(x) C \psi_{+,\,\vec{v}_j;\,j\beta}(x) \\ & -(L \to R) + \text{h.c.}\;. \end{aligned} \tag{26}$$

Because of the Riemann-Lebesgue theorem, in the $\mu \to \infty$ limit, the only non vanishing term in the sum corresponds to $\vec{v}_1 + \vec{v}_2 = 0$. Putting $\vec{v} = \vec{v}_1 = -\vec{v}_2$

and

$$\psi_{+,\,\pm\vec{v}_i;\,i\alpha}(x)\;\equiv\;\psi_{\pm\vec{v}_i;\,i\alpha}(x)\;, \tag{27}$$

we have:

$$\mathcal{L}_\Delta^{(s)}=-\frac{\Delta^{(s)}}{2}\,e^{2i\vec{q}\cdot\vec{x}}\sum_{\vec{v}}\epsilon_{ij}\epsilon^{\alpha\beta 3}e^{i\delta\mu\vec{v}\cdot\vec{x}}\psi_{+\vec{v};\,i\alpha}(x)C\psi_{-\vec{v};\,j\beta}(x)-(L\to R)+\text{h.c.}\;. \tag{28}$$

This term can be written as follows:

$$\mathcal{L}_\Delta^{(s)}\;=\;-\;\sum_{\vec{v}}\frac{\Delta^{(s)}_{\vec{q},\,\vec{v}}(\vec{x})}{2}\epsilon_{ij}\epsilon^{\alpha\beta 3}\psi_{+\vec{v};\,i\alpha}(x)C\psi_{-\vec{v};\,j\beta}(x)\;, \tag{29}$$

with

$$\Delta^{(s)}_{\vec{q},\,\vec{v}}(\vec{x})=\Delta^{(s)}e^{i\delta\mu\vec{v}\cdot\vec{x}}e^{2i\vec{q}\cdot\vec{x}}. \tag{30}$$

It is clear that this term violates translation and rotation invariance, since $\vec{q}$ is fixed. It also breaks colour symmetry from $SU(3)_c$ down to $SU(2)_c$.

The term corresponding to the vector condensate in the lagrangian can be written as follows:

$$\mathcal{L}_\Delta^{(v)}\;=\;-\sum_{\vec{v}}\frac{\Delta^{(v)}_{\vec{q},\,\vec{v}}(\vec{x})}{2}\sigma^1_{ij}\epsilon^{\alpha\beta 3}\psi_{+\vec{v};\,i\alpha}(x)C(\vec{\alpha}\cdot\vec{n})\psi_{-\vec{v};\,j\beta}(x)\;, \tag{31}$$

where

$$\Delta^{(v)}_{\vec{q},\,\vec{v}}(\vec{x})=\Delta^{(v)}e^{i\delta\mu\vec{v}\cdot\vec{x}}e^{2i\vec{q}\cdot\vec{x}}\;. \tag{32}$$

Here $\vec{n}=\vec{q}/|\vec{q}|$ is the direction corresponding to the total momentum carried by the Cooper pair. We observe that this term, besides colour symmetry, breaks translation, rotation invariance and flavor $SU(2)$ invariance as well.

In the basis introduced in the previous section, the effective lagrangian is

$$\begin{aligned}\mathcal{L}\;&=\;\mathcal{L}_0\;+\;\mathcal{L}_\Delta^{(s)}\;+\;\mathcal{L}_\Delta^{(v)}\;=\\ &=\;\frac{1}{2}\sum_{\vec{v}}\sum_{A=0}^{5}\chi^{A\,\dagger}\begin{pmatrix} i\delta_{AB}V\cdot\partial & \Delta_{AB}\\ \Delta_{AB} & i\delta_{AB}\tilde{V}\cdot\partial\end{pmatrix}\chi^B\;,\end{aligned} \tag{33}$$

with

$$\Delta_{AB}\;=\;\Delta^{(s)}_{AB}\;+\;(\vec{v}\cdot\vec{n})\;\Delta^{(v)}_{AB}\;. \tag{34}$$

Correspondingly, the effective action in momentum space reads:

$$S = \sum_{\vec{v}} \sum_{A,B=0}^{5} \frac{1}{2} \int \frac{d^4\ell}{(2\pi)^4} \frac{d^4\ell'}{(2\pi)^4} \chi^{A\dagger}(\ell') D^{-1}_{AB}(\ell',\ell) \chi_B(\ell) \ , \tag{35}$$

where $D^{-1}_{AB}(\ell',\ell)$ is the inverse propagator given by:

$$D^{-1}_{AB}(\ell',\ell) =$$
$$= \begin{pmatrix} V\cdot\ell\delta_{AB}\,\delta^4(\ell'-\ell) & \Delta_{AB}\,\delta^4(\ell'-\ell+\delta\mu\, v+2q) \\ \Delta_{AB}\,\delta^4(\ell'-\ell-\delta\mu\, v-2q) & \tilde{V}\cdot\ell\delta_{AB}\,\delta^4(\ell'-\ell) \end{pmatrix} .$$

Here we have defined $q^\mu = (0,\vec{q})$ and we have used $\psi_-^T C\vec{\alpha}\cdot\vec{n}\psi_+ = \vec{v}\cdot\vec{n}\psi_-^T C\psi_+$. The matrix $\Delta^{(s)}_{AB}$ is diagonal:

$$\Delta^{(s)}_{AB} = \delta_{AB}\Delta^{(s)}_A \tag{36}$$

with

$$\Delta^{(s)}_0 = \Delta^{(s)}, \qquad \Delta^{(s)}_1 = \Delta^{(s)}_2 = \Delta^{(s)}_3 = -\,\Delta^{(s)}, \qquad \Delta^{(s)}_4 = \Delta^{(s)}_5 = 0\ . \tag{37}$$

On the other hand one has

$$\Delta^{(v)}_{AB} \ = \ \Delta^{(v)} \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & i & 0 & 0 & 0 \\ 0 & -i & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} . \tag{38}$$

From these equations one can derive the quark propagator, given by

$$D_{AB}(\ell,\ell'') =$$
$$= \sum_C \begin{pmatrix} \tilde{V}\cdot\ell\,\delta_{AC}\,\delta^4(\ell-\ell'') & -\Delta_{AC}\,\delta^4(\ell-\ell''+\delta\mu\, v+2q) \\ -\Delta_{AC}\,\delta^4(\ell-\ell''-\delta\mu\, v-2q) & V\cdot\ell\,\delta_{AC}\,\delta^4(\ell-\ell'') \end{pmatrix} \times$$
$$\left((V\cdot\ell)(\tilde{V}\cdot\ell) - \mathbf{\Delta}^2\right)^{-1}_{CB} \tag{39}$$

with $\mathbf{\Delta}^2_{AB} = \sum_C \Delta_{AC}\Delta_{CB}$. One can observe that $\Delta_{4B} = \Delta_{5B} = \Delta_{A4} = \Delta_{A5} = 0$. Therefore the propagator for both the fields $\chi^{4,5}$ is simply given by

$$D(\ell,\ell') = \begin{pmatrix} (V\cdot\ell)^{-1}\delta^4(\ell-\ell') & 0 \\ 0 & (\tilde V\cdot\ell)^{-1}\,\delta^4(\ell-\ell') \end{pmatrix} . \tag{40}$$

As for the other four fields χ^A, $A = 0,\cdots,3$, we proceed as follows. Let us put

$$\Delta_0 = \Delta^{(s)}, \quad \Delta_1 = \vec v\cdot\vec n\,\Delta^{(v)} \ ; \tag{41}$$

we get

$$\Delta_{AB} = \begin{pmatrix} \Delta_0 & 0 & 0 & \Delta_1 \\ 0 & -\Delta_0 & i\Delta_1 & 0 \\ 0 & -i\Delta_1 & -\Delta_0 & 0 \\ \Delta_1 & 0 & 0 & -\Delta_0 \end{pmatrix} . \tag{42}$$

We note that the explicit appearance of the factor i in the Majorana mass matrix implies the breaking of charge conjugation according to its standard definition. However there is invariance under the following combined operations: take the complex conjugation, and send $\vec v \to -\vec v$, as it follows noticing that, correspondingly, $\Delta_1 \to -\Delta_1$.

The mass matrix can be made diagonal by going from the basis χ^A to the new basis $\tilde\chi^A$ defined as follows

$$\begin{aligned} \tilde\chi^0 &= \cos\theta\,\chi^0 + \sin\theta\,\chi^3 \\ \tilde\chi^1 &= \frac{1}{\sqrt 2}(\chi^1 - i\chi^2) \\ \tilde\chi^2 &= \frac{1}{\sqrt 2}(\chi^1 + i\chi^2) \\ \tilde\chi^3 &= -\sin\theta\,\chi^0 + \cos\theta\,\chi^3 \ , \end{aligned} \tag{43}$$

with $\tan\theta = \dfrac{\Delta_1}{\Delta_0 + \sqrt{\Delta_0^2 + \Delta_1^2}}$. In the new basis, the propagator for the fields $\tilde\chi^A$, $A = 0,\cdots,3$, is given by

$$D_{AB}(\ell,\ell') = \begin{pmatrix} \tilde V\cdot\ell\,\delta^4(\ell-\ell') & -M_A\delta^4(\ell-\ell'+\delta\mu v+2q) \\ -M_A\delta^4(\ell-\ell'-\delta\mu v-2q) & V\cdot\ell\,\delta^4(\ell-\ell') \end{pmatrix} \times \frac{\delta_{AB}}{(V\cdot\ell)(\tilde V\cdot\ell) - M_A^2} \tag{44}$$

where

$$
\begin{aligned}
M^A &= (M^0, M^1, M^2, M^3) = \\
&= \left(+\sqrt{\Delta_0^2+\Delta_1^2}, -(\Delta_0-\Delta_1), -(\Delta_0+\Delta_1), -\sqrt{\Delta_0^2+\Delta_1^2}\right) \; . \quad (45)
\end{aligned}
$$

One can note the mass differences between these states arising from the flavour symmetry term proportional to Δ_1. In absence of this term one would obtain $M^A = (+\Delta_0, -\Delta_0, -\Delta_0, -\Delta_0)$ corresponding to the states with isospin 0 (M^0) and isospin 1 (the remaining three).

4 Nambu-Goldstone Bosons and phonons

Let us consider again the breaking terms given by $\mathcal{L}_\Delta^{(s)}$ and $\mathcal{L}_\Delta^{(v)}$, that is

$$
\begin{aligned}
\mathcal{L}_\Delta &= \mathcal{L}_\Delta^{(s)} + \mathcal{L}_\Delta^{(v)} = \\
&= -\frac{1}{2}\, e^{2i\vec{q}\cdot\vec{x}} \sum_{\vec{v}} e^{i\delta\mu\vec{v}\cdot\vec{x}} \left[\Delta^{(s)}\epsilon_{ij} + \vec{v}\cdot\vec{n}\Delta^{(v)}\sigma^1_{ij}\right] \epsilon^{\alpha\beta 3}\psi_{+\vec{v};\,i\alpha}\, C\, \psi_{-\vec{v};\,j\beta} \\
&\quad -(L \to R) \; . \qquad (46)
\end{aligned}
$$

Since in the original theory this term corresponds to a condensate breaking spontaneously spatial translations and rotations and flavor $SU(2)_{L,R}$ symmetry, we will couple our effective lagrangian to the corresponding NGB fields treated as external sources.

To begin with, let us discuss isospin $SU(2)_L$ breaking, which arises from the symmetric flavour coupling σ^1_{ij} in $\mathcal{L}_\Delta^{(v)}$. We introduce $\vec{\eta} = (\eta_L^2, \eta_L^3)$ and the three-vector

$$
(\vec{I}_L)_i = I_{L,i}(\vec{\eta}) = (e^{i(\eta_L^2 S_2 + \eta_L^3 S_3)/g_L})_{i1} \qquad (47)
$$

with $\vec{S}$ the generators of the $SU(2)_L$ group in the isospin 1 representation, i.e.

$$
(S_i)_{jk} = -i\epsilon_{ijk} \; . \qquad (48)
$$

The fields η_L^2 and η_L^3 are the NGB fields associated to the breaking $SU(2)_L \to U(1)$. A simple calculation shows that:

$$
I_{L,1} = \cos\frac{|\vec{\eta}|}{g_L} = \sqrt{1-(I_{L,2})^2-(I_{L,3})^2}
$$

$$
\begin{aligned}
I_{L,2} &= -\frac{\eta_L^3}{|\vec{\eta}|}\sin\frac{|\vec{\eta}|}{g_L} \\
I_{L,3} &= +\frac{\eta_L^2}{|\vec{\eta}|}\sin\frac{|\vec{\eta}|}{g_L} \ .
\end{aligned}
\tag{49}
$$

It can be noted that

$$
|\vec{I}_L|^2 = 1 \tag{50}
$$

as it follows at once from the orthogonality of the rotation matrices. For small fields,

$$
\vec{I}_L \approx (1, -\frac{\eta_L^3}{g_L}, \frac{\eta_L^2}{g_L}) \ . \tag{51}
$$

The isospin symmetry is restated through the substitution in (46):

$$
\sigma_1 \to \vec{\sigma} \cdot \vec{I}_L \ . \tag{52}
$$

The same discussion goes through in the same way for the right-handed case.

Let us now consider spatial rotations. We take the z-axis pointing along the direction of $\vec{q}$. We define the three-vector

$$
(\vec{R})_i = R_i(\xi_1, \xi_2) = (e^{i(\xi_1 L_1 + \xi_2 L_2)/f_R})_{i3} \tag{53}
$$

with $\vec{L}$ the generators of the rotation group in the spin 1 representation, i.e.

$$
(L_i)_{jk} = -i\epsilon_{ijk} \tag{54}
$$

The fields $(\xi_1(x), \xi_2(x))$ are the NGB fields associated to the breaking $O(3) \to O(2)$ with $O(2)$ the residual rotational symmetry around the z-axis. It follows from the orthogonality of the rotation matrices that

$$
|\vec{R}|^2 = 1 \ . \tag{55}
$$

We obtain, by denoting $\vec{\xi} = (\xi_1, \xi_2)$,

$$
\begin{aligned}
R_1 &= -\frac{\xi_2}{|\vec{\xi}|}\sin\frac{|\vec{\xi}|}{f_R} \\
R_2 &= +\frac{\xi_1}{|\vec{\xi}|}\sin\frac{|\vec{\xi}|}{f_R}
\end{aligned}
$$

$$R_3 \quad = \quad \cos\frac{|\vec{\xi}|}{f_R} = \sqrt{1-R_1^2-R_2^2}\ , \tag{56}$$

and, for small fields,

$$\vec{R} \approx (-\frac{\xi_2}{f_R}, \frac{\xi_1}{f_R}, 1)\ . \tag{57}$$

Since R_i transforms as a vector under rotations, it follows that the symmetry is restored by the substitution:

$$\vec{v}\cdot\vec{n} = v_3 \to \vec{v}\cdot\vec{R}\ . \tag{58}$$

Let us finally consider the exponential factor $exp(2i\vec{q}\cdot\vec{x}) = exp(2iq\vec{n}\cdot\vec{x})$ in (46), which breaks both rotational and translational invariance. Notice that the factor $exp(i\delta\mu v\cdot x)$ does not break translation and rotation symmetries, since it comes from a field redefinition in a lagrangian which was originally invariant. As we have discussed, we can deal with rotations by the substitution $\vec{n}\to\vec{R}$, i.e. by the replacement:

$$e^{2iq\vec{n}\cdot\vec{x}} \to e^{2iq\vec{R}\cdot\vec{x}}\ . \tag{59}$$

As it stands, however, this factor still breaks the translational invariance, since, under a translation $\vec{x}\to\vec{x}\,' = \vec{x}+\vec{a}$ it transforms as

$$e^{2iq\vec{R}\cdot\vec{x}} \to e^{2iq\vec{R}\cdot\vec{x}} \times e^{2iq\vec{R}\cdot\vec{a}} \tag{60}$$

(as usual, under translations, $\vec{R}(x)\to\vec{R}'(x') = \vec{R}(x)$). Therefore we introduce a new field $U(x)$ with the transformation property under translation

$$U(x) \to U'(x') = e^{-2iq\vec{R}\cdot\vec{a}}U(x)\ . \tag{61}$$

In this way, under the transformation $x\to x+a$,

$$W(x) \equiv e^{2iq\vec{R}\cdot\vec{x}}U(x) \to W(x)\ , \tag{62}$$

and therefore the action, if expressed through this combination, is invariant under translations.

In order to reproduce correctly the original term $e^{2i\vec{q}\cdot\vec{x}}$ from the invariant expression

$$W(x) = e^{2iq\vec{R}\cdot\vec{x}}U(x)\ , \tag{63}$$

one has to require that, in the vacuum (the $LOFF$ state),

$$\langle U\rangle_0 = 1 \tag{64}$$

besides the above mentioned property $\langle\vec{R}\rangle_0 = (0,0,1)$. Therefore it is also possible to write the field $U(x)$ as

$$U(x) = e^{iT(x)/f_T} \tag{65}$$

where T is a scalar (phonon) field that, under translations, behaves as follows:

$$T(x) \to T(x) - 2qf_T\vec{a}\cdot\vec{R}\ . \tag{66}$$

Notice also that the theory depends only on the field $U(x)$ and as a consequence we have invariance under the transformation

$$T \to T + 2ni\pi f_T\,. \tag{67}$$

A different way to understand the field $T(x)$ is to introduce a vector field $\vec{T}(x)$ behaving under translations as $\vec{T} \to \vec{T} - 2f_T\vec{a}$. Then going to a moving frame defined by $T_\parallel \equiv \vec{T}\cdot\vec{R}$ and by the two orthogonal fields $\vec{T}_\perp = \vec{T} - \vec{R}T_\parallel$, we see that we can take $T = T_\parallel$. Since the only field interacting with the fermions is $T(x)$, the other two fields can be integrated away.

In conclusion, the interaction term, with the NGB fields treated as external sources, is

$$\begin{aligned}\mathcal{L}_{int} &= -\mathcal{L}_\Delta - \frac{1}{2}W(x)\sum_{\vec{v}} e^{i\delta\mu\vec{v}\cdot\vec{x}}\times \\ &\times \left[\Delta^{(s)}\epsilon_{ij} + \Delta^{(v)}(\vec{v}\cdot\vec{R})(\vec{\sigma}_{ij}\cdot\vec{I})\right]\epsilon^{\alpha\beta 3}\psi_{i,\alpha,\vec{v}}\,C\,\psi_{j,\beta,-\vec{v}} \\ &- (L\to R)\ .\end{aligned} \tag{68}$$

Notice that we have neglected the breaking of the colour symmetry, which has been considered elsewhere [9]. At the first order in the fields one gets the following three-linear couplings:

$$\mathcal{L}_{\xi\psi\psi} = -\frac{i}{2f_R}e^{2iqz}\sum_{\vec{v}} e^{i\delta\mu\vec{v}\cdot\vec{x}}\Big\{(y\xi_1 - x\xi_2)\left[\Delta^{(s)}\epsilon_{ij} + v_3\Delta^{(v)}\sigma^1_{ij}\right]$$

$$
\begin{aligned}
&+ \quad (v_2\xi_1 - v_1\xi_2)\Delta^{(v)}\sigma^1_{ij}\Big\}\epsilon^{\alpha\beta 3}\psi_{i,\alpha,\vec v}\,C\,\psi_{j,\beta,-\vec v} - (L \to R)\ ; \\
\mathcal{L}_{T\psi\psi} \quad &= \quad -\frac{iT}{2f_T}e^{2iqz}\sum_{\vec v} e^{i\delta\mu\vec v\cdot\vec x}\left[\Delta^{(s)}\epsilon_{ij} + v_3\Delta^{(v)}\sigma^1_{ij}\right]\epsilon^{\alpha\beta 3}\psi_{i,\alpha,\vec v}\,C\,\psi_{j,\beta,-\vec v} \\
&\quad -(L \to R)\ ; \\
\mathcal{L}_{\eta\psi\psi} \quad &= \quad \frac{1}{2f_I}e^{2iqz}\sum_{\vec v} e^{i\delta\mu\vec v\cdot\vec x}v_3\Delta^{(v)}\left[-\eta^3_L\sigma^2_{ij} + \eta^2_L\sigma^3_{ij}\right]\epsilon^{\alpha\beta 3}\psi_{i,\alpha,\vec v}\,C\,\psi_{j,\beta,-\vec v} \\
&\quad -(L \to R)\ . \qquad (69)
\end{aligned}
$$

Of course, we should also add the hermitian conjugated terms to the previous expressions. In general, a term of the form

$$
\psi^T C A \psi \ + \ h.c. \ = \ \sum_{\vec v}\psi_{i,\alpha,\vec v}CA_{ij,\alpha\beta,\vec v}\psi_{j,\beta,-\vec v} \ + \ h.c. \qquad (70)
$$

can be expressed in terms of the fields χ as follows

$$
\psi^T C A\psi + \psi^\dagger C^\dagger A^\dagger \psi^* = \chi^\dagger \begin{pmatrix} 0 & -A^\dagger \\ -A & 0 \end{pmatrix}\chi\ . \qquad (71)
$$

Through a bosonization procedure similar to the one employed in [12], one can derive an effective lagrangian for the NGB fields. Its general structure will be as follows:

$$
\mathcal{L}_{NGB} = \mathcal{L}_I + \mathcal{L}_R + \mathcal{L}_T + \mathcal{L}_{mixing}\ . \qquad (72)
$$

The first term can be generally written as follows

$$
\mathcal{L}_I = \frac{g_L^2}{2}\left(\dot{\vec I}_L\cdot\dot{\vec I}_L - v^2_{I_L}(\vec\nabla\vec I)\cdot(\vec\nabla\vec I_L)\right) + (L \to R)\ , \qquad (73)
$$

taking into account the breaking of the Lorentz invariance at finite density. For the second term we observe that there are various kinetic terms containing spatial derivatives that are parity conserving and invariant under rotations. We list only the terms that at the lowest order contribute to the kinetic lagrangian of the NGB $\vec\xi$ fields:

$$
\mathcal{L}_R \quad = \quad \frac{f_R^2}{2}\Big[\dot{\vec R}\cdot\dot{\vec R} - v_1^2\left(\partial_i R_k\right)\left(\partial_i R_k\right) - v_2^2\left(\partial_i R_i\right)^2 - v_3^2\left(R_i\partial_i R_k\right)\left(R_\ell\partial_\ell R_k\right)
$$

$$-v_4^2\left(\epsilon_{ijk}R_i\partial_j R_k\right)^2\Big] \ . \tag{74}$$

In order to write down the kinetic term for the phonon field, we observe that we have to satisfy translational invariance. To do that we can write

$$\mathcal{L}_T = \frac{f_T^2}{2}\left(\dot{W}\cdot\dot{W}^\dagger - v_T^2(\vec{\nabla}W)\cdot(\vec{\nabla}W^\dagger)\right) \ . \tag{75}$$

An equivalent way would be to introduce a covariant derivative operating on the field $U(x)$

$$D_\mu U(x) = (\partial_0, \partial_i + 2iqx_j\partial_i R_j)U(x) \ . \tag{76}$$

This term differs from $\vec{\nabla}W$ for a term proportional to $\vec{R}$. This would produce a difference in eq. (75) by a term $\vec{R}\cdot\vec{\nabla}W$. It is then invariant under translations and it can be added to the lagrangian as a mixing term

$$\mathcal{L}_{mixing} = \beta\vec{R}\cdot\vec{\nabla}W + h.c. \tag{77}$$

This concludes the construction of the effective Lagrangian $\mathcal{L}_{NGB}$.

5 Conclusions

We have constructed the effective lagrangian decribing the expected crystalline colour superconductive phase *(LOFF* phase) in high density and low temperature QCD with two massless flavors. The *LOFF* phase has been considered of possible astrophysical interest. It certainly deserves theoretical study as one of the components of the QCD phase diagram. The effective lagrangian we have constructed here extends previous results [12] to the crystalline phase. The construction is done by first integrating out the gluons and the negative energy (with respect to the Fermi sphere) fermions and describing positive energy fermions by velocity-dependent fields. The effective quarks behave as free fields with a Majorana mass term, breaking (besides colour, to be dealt with as in [9]) translational and rotational invariance, as well as $SU(2)_{L,R}$, due to the vector nature of one of the condensates [1]. The theory is then coupled to the relevant seven Nambu-Godstone Bosons: four from $SU(2)_{L,R}$ breaking, two from the breaking of rotational invariance, and one (the phonon) from the breaking of translational invariance. Finally one can integrate over the fermions leaving as dynamical fields only the Nambu-Goldstone bosons.

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